

# Existence and general stabilization of the Timoshenko system with a thermo-viscoelastic damping and a delay term in the internal feedback

Weican Zhou and Miaomiao Chen <sup>1</sup>

College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

**Abstract:** In this paper, we consider a Timoshenko system with a thermo-viscoelastic damping and a delay term in the internal feedback

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} + \int_0^t g(t-s)\theta_{xx}(s)ds \\ \quad + \mu_1\theta_t(x, t) + \mu_2\theta_t(x, t-\tau) = 0, & (x, t) \in (0, 1) \times (0, \infty) \end{cases}$$

together with initial datum and boundary conditions of Dirichlet type, where  $g$  is a positive non-increasing relaxation function and  $\mu_1, \mu_2$  are positive constants. Under an hypothesis between the weight of the delay term in the feedback and the the weight of the friction damping term, using the Faedo-Galerkin approximations together with some energy estimates, we prove the global existence of the solutions. Then, by introducing appropriate Lyapunov functionals, under the imposed constrain on the weights of the two feedbacks and the coefficients, we establish the general energy decay result from which the exponential and polynomial types of decay are only special cases.

**Keywords:** Global existence; General energy decay; Timoshenko system; Relaxation function.

**AMS Subject Classification (2010):** 35B35, 35L55, 74D05, 93D15

## 1 Introduction

In this paper we investigate the existence and decay properties of solutions for the Timoshenko system with a thermo-viscoelastic damping and a delay term of the form

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} + \int_0^t g(t-s)\theta_{xx}(s)ds \\ \quad + \mu_1\theta_t(x, t) + \mu_2\theta_t(x, t-\tau) = 0, & (x, t) \in (0, 1) \times (0, \infty) \end{cases} \quad (1.1)$$

---

<sup>1</sup>Corresponding author. E-mail: mmchennuist@163.com.

with the following initial datum and boundary conditions:

$$\begin{cases} \varphi(x, 0) = \varphi_0, & \varphi_t(x, 0) = \varphi_1, & \psi(x, 0) = \psi_0, & \psi_t(x, 0) = \psi_1, & x \in [0, 1], \\ \theta(x, 0) = \theta_0, & \theta_t(x, 0) = \theta_1, & & & x \in [0, 1], \\ \varphi(0, t) = \varphi(1, t) = \psi(0, t) = \psi(1, t) = \theta_x(0, t) = \theta_x(1, t) = 0, & t \in [0, \infty), \\ \theta_t(x, t - \tau) = f_0(x, t - \tau), & (x, t) \in (0, 1) \times (0, \tau), \end{cases} \quad (1.2)$$

where the coefficients  $\rho_1, \rho_2, \rho_3, K, b, \beta, \gamma, \delta, \mu_1$  and  $\mu_2$  are positive constants,  $\tau > 0$  represents the time delay.

System (1.1) arises in the theory of the transverse vibration of a beam which was first introduced by Timoshenko. In 1921, Timoshenko [25] considered the following system of coupled hyperbolic equations:

$$\begin{cases} \rho\varphi_{tt} - K(\varphi_x - \psi)_x = 0, & (x, t) \in (0, L) \times (0, \infty), \\ I_\rho\psi_{tt} - (EI\psi_x)_x - K(\varphi_x - \psi) = 0, & (x, t) \in (0, L) \times (0, \infty), \end{cases} \quad (1.3)$$

where  $\varphi$  is the transverse displacement of the beam and  $\psi$  is the rotation angle of the filament of the beam. The coefficients  $\rho, I_\rho, E, I$  and  $K$  are the density, the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus, respectively.

Many mathematicians have studied system (1.3) and some results concerning the existence and asymptotic behavior of solutions have been established, see for instance [2, 10, 22, 23] and the references therein. Kim and Renardy [5] considered (1.3) together with two linear boundary conditions of the form

$$\begin{aligned} K\psi(L, t) - K\frac{\partial\varphi}{\partial x}(L, t) &= \alpha\frac{\partial\varphi}{\partial t}(L, t), & t \in [0, \infty), \\ EI\frac{\partial\psi}{\partial x}(L, t) &= -\beta\frac{\partial\psi}{\partial t}(L, t), & t \in [0, \infty), \end{aligned}$$

and used the multiplier techniques to establish an exponential decay result for the energy of (1.3). They also provided numerical estimates to the eigenvalues of the operator associated with the system (1.3). Soufyane and Wehbe [24] considered

$$\begin{cases} \rho\varphi_{tt} - K(\varphi_x - \psi)_x = 0, & (x, t) \in (0, L) \times (0, \infty), \\ I_\rho\psi_{tt} - (EI\psi_x)_x - K(\varphi_x - \psi) + b(x)\psi_t = 0, & (x, t) \in (0, L) \times (0, \infty), \end{cases} \quad (1.4)$$

where  $b$  is a positive and continuous function, satisfying

$$b(x) \geq b_0 > 0, \quad \forall x \in [a_0, a_1] \subset [0, L].$$

They proved that the uniform stability of (1.4) holds if and only if the wave speeds are equal  $\left(\frac{K}{\rho} = \frac{EI}{I_\rho}\right)$ ; Otherwise only the asymptotic stability has been proved. For more results related to system (1.3), we refer the readers to [4, 16, 21, 18] and references therein.

In [13], Messaoudi and Said-Houari considered the one-dimensional linear Timoshenko system of thermoelastic type

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \beta\theta_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - \kappa\theta_{txx} = 0, & (x, t) \in (0, 1) \times (0, \infty). \end{cases} \quad (1.5)$$

They used the energy method to prove an exponential decay under the condition  $\frac{\rho_1}{K} = \frac{\rho_2}{b}$ . A similar result was also obtained by Rivera and Racke [17] and Messaoudi et al. [12]. Then, in [11], Messaoudi and Fareh also considered problem (1.5). By introducing the first and second-order energy functions, they proved a polynomial stability result under the condition  $\frac{\rho_1}{K} \neq \frac{\rho_2}{b}$ .

The case of time delay in the Timoshenko system has been studied by some authors. Said-Houari and Laskri [19] considered the following Timoshenko system with a constant time delay in the feedback:

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1 \psi_t + \mu_2 \psi_t(x, t - \tau) = 0, & (x, t) \in (0, 1) \times (0, \infty). \end{cases} \quad (1.6)$$

They established an exponential decay result for the case of equal-speed wave propagation ( $\frac{\rho_1}{K} = \frac{\rho_2}{b}$ ) under the assumption  $\mu_2 < \mu_1$ . Then, Kirane et al. [7] considered the Timoshenko system with a time-varying delay

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \mu_1 \psi_t + \mu_2 \psi_t(x, t - \tau(t)) = 0, & (x, t) \in (0, 1) \times (0, \infty), \end{cases}$$

where  $\tau(t) > 0$  represents the time varying delay,  $0 < \tau_0 \leq \tau(t) \leq \bar{\tau}$  and  $\mu_1, \mu_2$  are positive constants. Under the assumptions  $\frac{\rho_1}{K} = \frac{\rho_2}{b}$  and  $\mu_2 < \sqrt{1-d}\mu_1$ , where  $d$  is a constant such that  $\tau'(t) \leq d < 1$ , they proved that the energy decays exponentially.

In the presence of the thermo-viscoelastic damping, Djebabla and Tatar [1] considered the following Timoshenko system:

$$\begin{cases} \rho_1 \varphi_{tt} - K(\varphi_x + \psi)_x = 0, & (x, t) \in (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\varphi_x + \psi) + \gamma\theta_x = 0, & (x, t) \in (0, L) \times (0, \infty), \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \beta \int_0^t g(t-s)\theta_{xx}(s)ds + \gamma\psi_{ttx} = 0, & (x, t) \in (0, L) \times (0, \infty), \end{cases}$$

where  $\rho_1, \rho_2, \rho_3, K, b, \beta, \gamma$  and  $\delta$  are positive constants. They proved the exponential decay of solutions in the energy norm if and only if the coefficients satisfy  $\frac{b\rho_1}{K} - \rho_2 = \delta - \frac{K\rho_3}{\rho_1} = \gamma$  and  $g$  decays uniformly.

Kirane and Said-Houari [6] examined the system of viscoelastic wave equations with a linear

damping and a delay term

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_1 u_t(x,t) \\ \quad + \mu_2 u_t(x, t-\tau) = 0, & (x,t) \in \Omega \times (0, \infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0, \infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x, t-\tau) = f_0(x, t-\tau), & (x,t) \in \Omega \times (0, \tau), \end{cases} \quad (1.7)$$

where  $\Omega$  is a regular and bounded domain of  $\mathbb{R}^N$  ( $N \geq 1$ ),  $\mu_1, \mu_2$  are positive constants,  $\tau > 0$  represents the time delay and  $u_0, u_1, f_0$  are given functions belonging to suitable spaces. They proved that the energy of problem (1.7) decreases exponentially as  $t$  tends to infinity provided that  $0 < \mu_2 \leq \mu_1$  and  $g$  decays exponentially. Liu [9] considered the viscoelastic wave equation with a linear damping and a time-varying delay term in the feedback. Under suitable assumptions, he established a general decay result.

Motivated by above research, we consider the well-posedness and the general energy decay for problem (1.1). First, using the Faedo-Galerkin approximations together with some energy estimates, and under some restriction on the parameters  $\mu_1$  and  $\mu_2$ , the system is showed to be well-posed. Then, under the hypothesis  $\mu_2 \leq \mu_1$ , we prove a general decay of the total energy of our problem by using energy method. Our method of proof uses some ideas developed in [6] for the wave equation with a viscoelastic damping and a delay term, enabling us to obtain suitable Lyapunov functionals, from which we derive the desired results. We recall that for  $\mu_1 = \mu_2$ , Nicaise and Pignotti showed in [14] that some instabilities may occur. Here, due to the presence of the viscoelastic damping, we prove that our energy still decays generally even if  $\mu_1 = \mu_2$ .

The remaining part of this paper is organized as follows. In Section 2, we present some materials and recall some useful lemmas needed for our work and state our main results. In Section 3, we will prove the well-posedness of the solution. We will prove several technical lemmas and the general decay result under the two cases:  $\mu_2 < \mu_1$  and  $\mu_2 = \mu_1$  in Section 4.

## 2 Preliminaries and main results

In this section, we present some assumptions and state the main results. We use the standard Lebesgue space  $L^2(0,1)$  and the Sobolev space  $H_0^1(0,1)$  with their usual scalar products and norms and define the following space  $X$  as

$$X = [H_0^1(0,1) \times L^2(0,1)]^2 \times V \times H,$$

where

$$V = H^1(0,1) \cap H$$

and

$$H = \{\theta \in L^2(0, 1) | \theta_x(0, t) = \theta_x(1, t) = 0, \forall t \in [0, \infty)\}.$$

First, in order to exhibit the dissipative nature of system (1.1), as in [13], we introduce the new variables  $\Phi = \varphi_t$  and  $\Psi = \psi_t$ . Then, as in [15], we introduce the function

$$z(x, \rho, t) = \theta_t(x, t - \rho\tau), \quad (x, \rho, t) \in (0, 1) \times (0, 1) \times (0, \infty).$$

Then, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad (x, \rho, t) \in (0, 1) \times (0, 1) \times (0, \infty).$$

Therefore, problem (1.1) is equivalent to

$$\begin{cases} \rho_1 \Phi_{tt} - K(\Phi_x + \Psi)_x = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_2 \Psi_{tt} - b\Psi_{xx} + K(\Phi_x + \Psi) + \beta\theta_{tx} = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \gamma\Psi_{tx} + \int_0^t g(t-s)\theta_{xx}(s)ds \\ \quad + \mu_1\theta_t(x, t) + \mu_2 z(x, 1, t) = 0, & (x, t) \in (0, 1) \times (0, \infty), \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, & (x, \rho, t) \in (0, 1) \times (0, 1) \times (0, \infty) \end{cases} \quad (2.1)$$

with the following initial datum and boundary conditions:

$$\begin{cases} \Phi(x, 0) = \Phi_0, \quad \Phi_t(x, 0) = \Phi_1, \quad \Psi(x, 0) = \Psi_0, \quad \Psi_t(x, 0) = \Psi_1, & x \in [0, 1], \\ \theta(x, 0) = \theta_0, \quad \theta_t(x, 0) = \theta_1, & x \in [0, 1], \\ \Phi(0, t) = \Phi(1, t) = \Psi(0, t) = \Psi(1, t) = \theta_x(0, t) = \theta_x(1, t) = 0, & t \in [0, \infty), \\ z(x, 0, t) = \theta_t(x, t), & (x, t) \in (0, 1) \times (0, \infty), \\ \theta_t(x, t - \tau) = f_0(x, t - \tau), & (x, t) \in (0, 1) \times (0, \tau). \end{cases} \quad (2.2)$$

Next, we denote by  $*$  the usual convolution term

$$(g * h)(t) = \int_0^t g(t-s)h(s)ds$$

and the binary operators  $\diamond$  and  $\circ$ , respectively, by

$$(g \diamond h)(t) = \int_0^t g(t-s) (h(t) - h(s)) ds$$

and

$$(g \circ h)(t) = \int_0^t g(t-s) (h(t) - h(s))^2 ds.$$

The following lemma was introduced in [6]. It will be used in Section 4 to prove the general energy decay result for problem (1.1)-(1.2).

**Lemma 2.1** For any function  $g \in C^1(\mathbb{R})$  and any  $h \in H^1(0, 1)$ , we have

$$(g * h)(t)h_t(t) = -\frac{1}{2}g(t)|h(t)|^2 + \frac{1}{2}(g' \diamond h)(t) - \frac{1}{2} \frac{d}{dt} \left\{ (g \diamond h)(t) - \left( \int_0^t g(s)ds \right) |h(t)|^2 \right\}. \quad (2.3)$$

The proof of this lemma follows by differentiating the term  $g \diamond h$ .

**Lemma 2.2** ([1]) For any function  $g \in C([0, \infty), \mathbb{R}_+)$  and any  $h \in L^2(0, 1)$ , we have

$$[(g \diamond h)(t)]^2 \leq \left( \int_0^t g(s)ds \right) (g \circ h)(t), \quad t \geq 0. \quad (2.4)$$

Now, we assume that the kernel  $g$  satisfies the following assumptions:

(H1)  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a differential function such that

$$g(0) > 0, \quad \lambda = \delta - \int_0^\infty g(s)ds = \delta - \bar{g} > 0;$$

(H2) There exists a non-increasing differential function  $\zeta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying

$$g'(t) \leq -\zeta(t)g(t), \quad t \geq 0$$

and

$$\int_0^{+\infty} \zeta(t)dt = +\infty.$$

To state our decay result, we introduce the energy functional associated to problem (2.1)

$$E(t) = \frac{\gamma}{2} \int_0^1 \{ \rho_1 \Phi_t^2 + \rho_2 \Psi_t^2 + K |\Phi_x + \Psi|^2 + b \Psi_x^2 \} dx + \frac{\beta}{2} \int_0^1 \left\{ \rho_3 \theta_t^2 + \left( \delta - \int_0^t g(s)ds \right) \theta_x^2 + (g \circ \theta_x) + \xi \int_0^1 z^2(x, \rho, t) d\rho \right\} dx, \quad (2.5)$$

where  $\xi$  is a positive constant such that

$$\tau \mu_2 < \xi < \tau(2\mu_1 - \mu_2) \quad \text{if } \mu_2 < \mu_1, \quad (2.6)$$

$$\xi = \tau \mu_2 \quad \text{if } \mu_2 = \mu_1. \quad (2.7)$$

Our main results read as follows.

**Theorem 2.3** Assume that  $\mu_2 \leq \mu_1$ . Then for any given  $(\Phi_0, \Phi_1, \Psi_0, \Psi_1, \theta_0, \theta_1) \in X$ ,  $f_0 \in L^2((0, 1) \times (0, 1))$  and  $T > 0$ , there exists a unique weak solution  $(\Phi, \Psi, \theta, z)$  of problem (2.1)-(2.2) on  $(0, T)$  such that

$$(\Phi, \Psi, \theta) \in C\left([0, T], [H_0^1(0, 1)]^2 \times V\right) \cap C^1\left([0, T], [L^2(0, 1)]^2 \times H\right).$$

**Theorem 2.4** Assume that  $\mu_2 \leq \mu_1$  and  $g$  satisfies (H1) and (H2). Assume further that initial datum satisfy

$$(\Phi_0, \Phi_1, \Psi_0, \Psi_1, \theta_0, \theta_1) \in X, \quad f_0 \in L^2((0, 1) \times (0, 1)) \quad (2.8)$$

and the coefficients  $\rho_1, \rho_2, \rho_3, K, b, \gamma, \delta$  and  $\beta$  satisfy

$$\frac{b\rho_1}{K} - \rho_2 = \gamma, \quad \delta - \frac{K\rho_3}{\rho_1} = \beta. \quad (2.9)$$

Then for any  $t_0 > 0$ , there exist two positive constants  $A$  and  $\omega$  independent of the initial datum such that

$$E(t) \leq Ae^{-\omega \int_{t_0}^t \zeta(s) ds}, \quad t \geq t_0. \quad (2.10)$$

### 3 Proof of Theorem 2.3

In this section, we will use the Faedo-Galerkin approximations together with some energy estimates, to prove the existence of the unique solution of problem (2.1)-(2.2) as stated in Theorem 2.3.

**Proof.** We divide the proof of Theorem 2.3 in two steps: the construction of approximations and then thanks to certain energy estimates, we pass to the limit.

#### Step 1: Faedo-Galerkin approximations.

As in [6] and [3], we construct approximations of the solution  $(\Phi, \Psi, \theta, z)$  by the Faedo-Galerkin method as follows. For every  $n \geq 1$ , let  $W_n = \text{span}\{\omega_1, \dots, \omega_n\}$  be a Hilbert basis of the space  $H_0^1(0, 1)$ .

Now, we define for  $1 \leq j \leq n$  the sequence  $\bar{\varphi}_j(x, \rho)$  as follows

$$\bar{\varphi}_j(x, 0) = \omega_j(x).$$

Then, we may extend  $\bar{\varphi}_j(x, 0)$  by  $\bar{\varphi}_j(x, \rho)$  over  $L^2((0, 1) \times [0, 1])$  and denote  $V_n = \text{span}\{\bar{\varphi}_1, \dots, \bar{\varphi}_n\}$ . We choose sequences  $(\Phi_{0n}, \Psi_{0n}, \theta_{0n})$  and  $(\Phi_{1n}, \Psi_{1n}, \theta_{1n})$  in  $W_n$  and a sequence  $(z_{0n})$  in  $V_n$  such that  $(\Phi_{0n}, \Phi_{1n}, \Psi_{0n}, \Psi_{1n}, \theta_{0n}, \theta_{1n}) \rightarrow (\Phi_0, \Phi_1, \Psi_0, \Psi_1, \theta_0, \theta_1)$  strongly in  $X$  and  $z_{0n} \rightarrow f_0$  strongly in  $L^2((0, 1) \times (0, 1))$ .

We define now the approximations:

$$(\Phi_n(x, t), \Psi_n(x, t), \theta_n(x, t)) = \sum_{j=1}^n (f_{jn}(t), y_{jn}(t), h_{jn}(t)) \omega_j(x) \quad (3.1)$$

and

$$z_n(x, \rho, t) = \sum_{j=1}^n l_{jn}(t) \bar{\varphi}_j(x, \rho), \quad (3.2)$$

where  $(\Phi_n(t), \Psi_n(t), \theta_n(t), z_n(t))$  satisfies the following problem:

$$\left\{ \begin{array}{l} \rho_1 \int_0^1 \Phi_{ttn} \omega_j dx + K \int_0^1 (\Phi_{xn} + \Psi_n) \omega_{xj} dx = 0, \\ \rho_2 \int_0^1 \Psi_{ttn} \omega_j dx + b \int_0^1 \Psi_{xn} \omega_{xj} dx + K \int_0^1 (\Phi_{xn} + \Psi_n) \omega_j dx - \beta \int_0^1 \theta_{tn} \omega_{xj} dx = 0, \\ \rho_3 \int_0^1 \theta_{ttn} \omega_j dx + \delta \int_0^1 \theta_{xn} \omega_{xj} dx - \gamma \int_0^1 \Psi_{tn} \omega_{xj} dx - \int_0^t g(t-s) \int_0^1 \theta_{xn}(s) \omega_{xj} dx ds \\ \quad + \int_0^1 (\mu_1 \theta_{tn}(x, t) + \mu_2 z_n(x, 1, t)) \omega_j dx = 0, \\ (\Phi_n(0), \Psi_n(0), \theta_n(0)) = (\Phi_{0n}, \Psi_{0n}, \theta_{0n}), \\ (\Phi_{tn}(0), \Psi_{tn}(0), \theta_{tn}(0)) = (\Phi_{1n}, \Psi_{1n}, \theta_{1n}), \\ z_n(x, 0, t) = \theta_{tn}(x, t) \end{array} \right. \quad (3.3)$$

and

$$\left\{ \begin{array}{l} \int_0^1 (\tau z_{tn}(x, \rho, t) + z_{\rho n}(x, \rho, t)) \bar{\varphi}_j dx = 0, \\ z_n(x, \rho, 0) = z_{0n}, \end{array} \right. \quad (3.4)$$

for  $1 \leq j \leq n$ . According to the standard theory of ordinary differential equations, the finite dimensional problem (3.3)-(3.4) has a solution  $(f_{jn}(t), y_{jn}(t), h_{jn}(t), l_{jn}(t))_{j=1, \dots, n}$  defined on  $[0, t_n)$ . Then a priori estimates that follow imply that in fact  $t_n = T$ .

### Step 2: Energy estimates.

Multiplying Eq. (2.1)<sub>1</sub> by  $\gamma f'_{jn}$ , (2.1)<sub>2</sub> by  $\gamma y'_{jn}$  and (2.1)<sub>3</sub> by  $\beta h'_{jn}$ , integrating over  $(0, 1)$  using integration by parts and Lemma 2.1, we get, for every  $n \geq 1$ ,

$$\begin{aligned} & \frac{\gamma}{2} [\rho_1 \|\Phi_{tn}\|_2^2 + \rho_2 \|\Psi_{tn}\|_2^2 + K \|\Phi_{xn} + \Psi_n\|_2^2 + b \|\Psi_{xn}\|_2^2] \\ & + \frac{\beta}{2} \left[ \rho_3 \|\theta_{tn}\|_2^2 + \left( \delta - \int_0^t g(s) ds \right) \|\theta_{xn}\|_2^2 + (g \circ \theta_{xn}) \right] \\ & + \beta \mu_1 \int_0^1 \|\theta_{tn}(s)\|_2^2 ds + \beta \mu_2 \int_0^t \int_0^1 \theta_{tn}(x, s) z_n(x, 1, s) dx ds + \frac{\beta}{2} \int_0^t g(s) \|\theta_{xn}(s)\|_2^2 ds \\ & - \frac{\beta}{2} \int_0^t (g' \circ \theta_{xn})(s) ds \\ & = \frac{\gamma}{2} [\rho_1 \|\Phi_1\|_2^2 + \rho_2 \|\Psi_1\|_2^2 + K \|\Phi_{x0} + \Psi_0\|_2^2 + b \|\Psi_{x0}\|_2^2] + \frac{\beta}{2} [\rho_3 \|\theta_1\|_2^2 + \delta \|\theta_{x0}\|_2^2]. \end{aligned} \quad (3.5)$$

Let  $\xi > 0$  to be chosen later. Multiplying Eq. (2.1)<sub>4</sub> by  $\frac{\xi}{\tau} l'_{jn}(t)$  integrating over  $(0, t) \times (0, 1)$ , we obtain

$$\begin{aligned} & \frac{\xi}{2} \int_0^1 \int_0^1 z_n^2(x, \rho, t) d\rho dx + \frac{\xi}{\tau} \int_0^t \int_0^1 \int_0^1 z_{n\rho} z_n(x, \rho, s) d\rho dx ds \\ & = \frac{\xi}{2} \|z_{0n}\|_{L^2((0,1) \times (0,1))}^2. \end{aligned} \quad (3.6)$$



Now, to handle the last term in the left-hand side of (3.6), we remark that

$$\begin{aligned} \int_0^t \int_0^1 \int_0^1 z_{n\rho} z_n(x, \rho, t) d\rho dx ds &= \frac{1}{2} \int_0^t \int_0^1 \int_0^1 \frac{\partial}{\partial \rho} z_n^2(x, \rho, s) d\rho dx ds \\ &= \frac{1}{2} \int_0^t \int_0^1 (z_n^2(x, 1, s) - z_n^2(x, 0, s)) dx ds. \end{aligned} \quad (3.7)$$

Summing up the identities (3.5) and (3.6) and taking into account (3.7), we get

$$\begin{aligned} \mathcal{E}_n(t) &+ \beta \left( \mu_1 - \frac{\xi}{2\tau} \right) \int_0^t \|\theta_{tn}\|_2^2 ds + \frac{\xi\beta}{2\tau} \int_0^t \int_0^1 z_n^2(x, 1, s) dx ds \\ &+ \beta\mu_2 \int_0^t \int_0^1 z_n(x, 1, s) \theta_{tn}(x, s) ds \\ &+ \frac{\beta}{2} \int_0^t g(s) \|\theta_{xn}(s)\|_2^2 ds - \frac{\beta}{2} \int_0^t (g' \circ \theta_{xn})(s) ds \\ &= \mathcal{E}_n(0), \end{aligned} \quad (3.8)$$

where

$$\begin{aligned} \mathcal{E}_n(t) &= \frac{\gamma}{2} [\rho_1 \|\Phi_{tn}\|_2^2 + \rho_2 \|\Psi_{tn}\|_2^2 + K \|\Phi_{xn} + \Psi_n\|_2^2 + b \|\Psi_{xn}\|_2^2] \\ &+ \frac{\beta}{2} \left[ \rho_3 \|\theta_{tn}\|_2^2 + \left( \delta - \int_0^t g(s) ds \right) \|\theta_{xn}\|_2^2 + (g \circ \theta_{xn}) + \xi \|z_n\|_{L^2((0,1) \times (0,1))}^2 \right]. \end{aligned} \quad (3.9)$$

At this point, we have to distinguish the following two cases:

Case 1: We suppose that  $\mu_2 < \mu_1$ . Let us choose  $\xi$  satisfies inequality (2.6). Using Young's inequality, (3.8) leads to:

$$\begin{aligned} \mathcal{E}_n(t) &+ \beta \left( \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_0^t \|\theta_{tn}\|_2^2 ds + \beta \left( \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_0^t \int_0^1 z_n^2(x, 1, s) dx ds \\ &+ \frac{\beta}{2} \int_0^t g(s) \|\theta_{xn}(s)\|_2^2 ds - \frac{\beta}{2} \int_0^t (g' \circ \theta_{xn})(s) ds \\ &\leq \mathcal{E}_n(0). \end{aligned}$$

Consequently, using (2.7), we can find two positive constants  $c_1$  and  $c_2$  such that

$$\begin{aligned} \mathcal{E}_n(t) &+ c_1 \int_0^t \|\theta_{tn}\|_2^2 ds + c_2 \int_0^t \int_0^1 z_n^2(x, 1, s) dx ds \\ &+ \frac{\beta}{2} \int_0^t g(s) \|\theta_{xn}(s)\|_2^2 ds - \frac{\beta}{2} \int_0^t (g' \circ \theta_{xn})(s) ds \\ &\leq \mathcal{E}_n(0). \end{aligned} \quad (3.10)$$

Case 2: We suppose that  $\mu_2 = \mu_1 = \mu$  and choose  $\xi = \tau\mu$ . Then, inequality (3.10) takes the form

$$\mathcal{E}_n(t) + \frac{\beta}{2} \int_0^t g(s) \|\theta_{xn}(s)\|_2^2 ds - \frac{\beta}{2} \int_0^t (g' \circ \theta_{xn})(s) ds \leq \mathcal{E}_n(0). \quad (3.11)$$

Now, in both cases and since the sequences  $(\Phi_{0n})_{n \in \mathbb{N}}$ ,  $(\Phi_{1n})_{n \in \mathbb{N}}$ ,  $(\Psi_{0n})_{n \in \mathbb{N}}$ ,  $(\Psi_{1n})_{n \in \mathbb{N}}$ ,  $(\theta_{0n})_{n \in \mathbb{N}}$ ,  $(\theta_{1n})_{n \in \mathbb{N}}$  and  $(z_{0n})_{n \in \mathbb{N}}$  converge, and using (H1) and (H2), we can find a positive constant  $C$  independent of  $n$  such that

$$\mathcal{E}_n(t) \leq C. \quad (3.12)$$

Therefore, using the fact that  $\delta - \int_0^1 g(s)ds \geq \lambda$ , the last estimate (3.12) together with (3.9) give us, for all  $n \in \mathbb{N}$ ,  $t_n = T$ , we deduce that

$$(\Phi_n, \Psi_n, \theta_n)_{n \in \mathbb{N}} \text{ is bounded in } L^\infty \left( 0, T; [H_0^1(0, 1)]^2 \times V \right), \quad (3.13)$$

$$(\Phi_{tn}, \Psi_{tn}, \theta_{tn})_{n \in \mathbb{N}} \text{ is bounded in } L^\infty \left( 0, T; [L^2(0, 1)]^2 \times H \right) \quad (3.14)$$

and

$$(z_n)_{n \in \mathbb{N}} \text{ is bounded in } L^\infty \left( 0, T; L^2((0, 1) \times (0, 1)) \right). \quad (3.15)$$

Consequently, we may conclude that

$$\begin{aligned} (\Phi_n, \Psi_n, \theta_n) &\rightharpoonup (\Phi, \Psi, \theta) \text{ weak}^* && \text{in } L^\infty \left( 0, T; [H_0^1(0, 1)]^2 \times V \right), \\ (\Phi_{tn}, \Psi_{tn}, \theta_{tn}) &\rightharpoonup (\Phi_t, \Psi_t, \theta_t) \text{ weak}^* && \text{in } L^\infty \left( 0, T; [L^2(0, 1)]^2 \times H \right), \\ z_n &\rightharpoonup z \text{ weak}^* && \text{in } L^\infty \left( 0, T; L^2((0, 1) \times (0, 1)) \right). \end{aligned} \quad (3.16)$$

From (3.13), (3.14) and (3.15), we have  $(\Phi_n, \Psi_n, \theta_n)_{n \in \mathbb{N}}$  is bounded in  $L^\infty \left( 0, T; [H_0^1(0, 1)]^2 \times V \right)$ . Then,  $(\Phi_n, \Psi_n, \theta_n)_{n \in \mathbb{N}}$  is bounded in  $L^2 \left( 0, T; [H_0^1(0, 1)]^2 \times V \right)$ . Since  $(\Phi_{tn}, \Psi_{tn}, \theta_{tn})_{n \in \mathbb{N}}$  is bounded in  $L^\infty \left( 0, T; [L^2(0, 1)]^2 \times H \right)$ ,  $(\Phi_{tn}, \Psi_{tn}, \theta_{tn})_{n \in \mathbb{N}}$  is bounded in  $L^2 \left( 0, T; [L^2(0, 1)]^2 \times H \right)$ . Consequently,  $(\Phi_n, \Psi_n, \theta_n)_{n \in \mathbb{N}}$  is bounded in  $H^1 \left( 0, T; [H^1(0, 1)]^2 \times V \right)$ .

Since the embedding  $H^1 \left( 0, T; [H^1(0, 1)]^2 \times V \right) \hookrightarrow L^2 \left( 0, T; [L^2(0, 1)]^2 \times H \right)$  is compact, using Aubin-Lions theorem [8], we can extract subsequences  $(\Phi_\mu, \Psi_\mu, \theta_\mu)_{\mu \in \mathbb{N}}$  of  $(\Phi_n, \Psi_n, \theta_n)_{n \in \mathbb{N}}$  such that

$$(\Phi_\mu, \Psi_\mu, \theta_\mu) \rightharpoonup (\Phi, \Psi, \theta) \text{ strongly in } L^2 \left( 0, T; [L^2(0, 1)]^2 \times H \right).$$

Therefore,

$$(\Phi_\mu, \Psi_\mu, \theta_\mu) \rightharpoonup (\Phi, \Psi, \theta) \text{ strongly and a.e on } (0, T) \times (0, 1).$$

The proof now can be completed arguing as in [8].

## 4 Proof of Theorem 2.4

In this section, under the hypothesis  $\mu_2 \leq \mu_1$ , we show that the energy of the solution of problem (2.1)-(2.2) decreases generally as  $t$  tends to infinity by using the energy method and suitable Lyapunov functionals. We will separately discuss the two cases which are the case  $\mu_2 < \mu_1$  and the case  $\mu_2 = \mu_1$  since the proofs are slightly different.

### 4.1 The case $\mu_2 < \mu_1$

Our goal now is to prove the above energy  $E(t)$  is a non-increasing functional along the trajectories. More precisely, we have the following result:

**Lemma 4.1** *Suppose that (H1) and (H2) hold and let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2). Then we have*

$$\begin{aligned} E'(t) &\leq -\frac{\beta}{2}g(t) \int_0^1 \theta_x^2 dx + \frac{\beta}{2} \int_0^1 (g' \circ \theta_x) dx - \beta \left( \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_0^1 \theta_t^2 dx \\ &\quad - \beta \left( \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \int_0^1 z^2(x, 1, t) dx \leq 0, \quad t \geq 0. \end{aligned} \quad (4.1)$$

**Proof.** Multiplying Eq. (2.1)<sub>1</sub> by  $\gamma\Phi_t$ , (2.1)<sub>2</sub> by  $\gamma\Psi_t$  and (2.1)<sub>3</sub> by  $\beta\theta_t$ , integrating over  $(0, 1)$  and performing an integration by parts, we find

$$\begin{aligned} &\frac{\gamma}{2} \frac{d}{dt} \left[ \int_0^1 \{ \rho_1 \Phi_t^2 + \rho_2 \Psi_t^2 + K |\Phi_x + \Psi|^2 + b \Psi_x^2 \} dx \right] \\ &+ \frac{\beta}{2} \frac{d}{dt} \left[ \int_0^1 \left\{ \rho_3 \theta_t^2 + \left( \delta - \int_0^t g(\tau) d\tau \right) \theta_x^2 + (g \circ \theta_x) \right\} dx \right] \\ &= -\beta\mu_1 \int_0^1 \theta_t^2 dx - \beta\mu_2 \int_0^1 \theta_t z(x, 1, t) dx - \frac{\beta}{2} g(t) \int_0^1 \theta_x^2 dx + \frac{\beta}{2} \int_0^1 (g' \circ \theta_x) dx. \end{aligned} \quad (4.2)$$

Now, multiplying the Eq. (2.1)<sub>4</sub> by  $\frac{\xi}{\tau}z$ , integrating the result over  $(0, 1) \times (0, 1)$  with respect to  $\rho$  and  $x$ , respectively, we obtain

$$\begin{aligned} \frac{\xi}{2} \frac{d}{dt} \int_0^1 \int_0^1 z^2(x, \rho, t) d\rho dx &= -\frac{\xi}{\tau} \int_0^1 \int_0^1 z z_\rho(x, \rho, t) d\rho dx \\ &= -\frac{\xi}{2\tau} \int_0^1 \int_0^1 \frac{\partial}{\partial \rho} z^2(x, \rho, t) d\rho dx \\ &= \frac{\xi}{2\tau} \int_0^1 [z^2(x, 0, t) - z^2(x, 1, t)] dx \\ &= \frac{\xi}{2\tau} \int_0^1 (\theta_t^2 - z^2(x, 1, t)) dx. \end{aligned} \quad (4.3)$$

By Young's inequality, we have

$$-\mu_2 \int_0^1 \theta_t z(x, 1, t) dx \leq \frac{\mu_2}{2} \int_0^1 \theta_t^2 dx + \frac{\mu_2}{2} \int_0^1 z^2(x, 1, t) dx. \quad (4.4)$$

Then, exploiting (4.3) and (4.4) our conclusion holds.  $\square$

Now we are going to construct a Lyapunov functional  $\mathcal{L}$  equivalent to  $E$ . For this, we define several functionals which allow us to obtain the needed estimates. We introduce the multiplier  $w$  given by the solution of the Dirichlet problem

$$-w_{xx} = \Psi_x, \quad w(0) = w(1) = 0 \quad (4.5)$$

and define the functional

$$I_1(t) = \int_0^1 (\rho_2 \Psi_t \Psi + \rho_1 \Phi_t w + \beta \theta_x \Psi) dx, \quad t \geq 0. \quad (4.6)$$

The derivative of this functional will provide us with the term

$$- \int_0^1 \Psi_x^2 dx.$$

**Lemma 4.2** *Let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2). For any  $\varepsilon_1 > 0$ , we have*

$$I_1'(t) \leq -b \int_0^1 \Psi_x^2 dx + \left( \frac{3\rho_2}{2} + \frac{\rho_1^2 C_p}{4\varepsilon_1} \right) \int_0^1 \Psi_t^2 dx + \varepsilon_1 \int_0^1 \Phi_t^2 dx + \frac{\beta^2}{2\rho_2} \int_0^1 \theta_x^2 dx, \quad (4.7)$$

where  $C_p$  is the Poincaré constant.

**Proof.** By differentiating  $I_1$  with respect to  $t$  and using Eqs. (2.1) and (2.2) we conclude that

$$I_1'(t) = -b \int_0^1 \Psi_x^2 dx + \rho_2 \int_0^1 \Psi_t^2 dx - K \int_0^1 \Psi^2 dx + K \int_0^1 w_x^2 dx + \rho_1 \int_0^1 \Phi_t w_t dx + \beta \int_0^1 \theta_x \Psi_t dx.$$

By exploiting the inequalities

$$\int_0^1 w_x^2 dx \leq \int_0^1 \Psi^2 dx \leq C_p \int_0^1 \Psi_x^2 dx,$$

$$\int_0^1 w_t^2 dx \leq C_p \int_0^1 w_{tx}^2 dx \leq C_p \int_0^1 \Psi_t^2 dx$$

and Young's inequality, we find that

$$I_1'(t) \leq -b \int_0^1 \Psi_x^2 dx + \frac{3\rho_2}{2} \int_0^1 \Psi_t^2 dx + \frac{\beta^2}{2\rho_2} \int_0^1 \theta_x^2 dx + \varepsilon_1 \int_0^1 \Phi_t^2 dx + \frac{\rho_1^2}{4\varepsilon_1} \int_0^1 w_t^2 dx.$$

Thus,

$$I_1'(t) \leq -b \int_0^1 \Psi_x^2 dx + \left( \frac{3\rho_2}{2} + \frac{\rho_1^2 C_p}{4\varepsilon_1} \right) \int_0^1 \Psi_t^2 dx + \varepsilon_1 \int_0^1 \Phi_t^2 dx + \frac{\beta^2}{2\rho_2} \int_0^1 \theta_x^2 dx,$$

which is exactly (4.7).  $\square$

In order to obtain the negative term of  $\int_0^1 \theta_x^2 dx$ , we define the functional

$$I_2(t) = \int_0^1 \left( \rho_3 \theta_t \theta + \gamma \Psi_x \theta + \frac{\mu_1}{2} \theta^2 \right) dx.$$

**Lemma 4.3** *Let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2). For any  $\varepsilon_2 > 0$ , we have*

$$\begin{aligned} I_2'(t) \leq & -\frac{\lambda}{2} \int_0^1 \theta_x^2 dx + \left( \rho_3 + \frac{\gamma^2}{4\varepsilon_2} \right) \int_0^1 \theta_t^2 dx + \varepsilon_2 \int_0^1 \Psi_x^2 dx \\ & + \frac{1}{\lambda} \int_0^t g(s) ds \int_0^1 (g \circ \theta_x) dx + \frac{\mu_2^2 C_p}{\lambda} \int_0^1 z^2(x, 1, t) dx. \end{aligned} \quad (4.8)$$

**Proof.** A simple differentiation leads to

$$I_2'(t) = \rho_3 \int_0^1 \theta_t^2 dx + \rho_3 \int_0^1 \theta_{tt} \theta dx + \gamma \int_0^1 \Psi_{tx} \theta dx + \gamma \int_0^1 \Psi_x \theta_t dx + \mu_1 \int_0^1 \theta_t \theta dx.$$

By using Eq. (2.1)<sub>3</sub> and (2.2), we arrive at

$$I_2'(t) = - \left( \delta - \int_0^t g(s) ds \right) \int_0^1 \theta_x^2 dx + \rho_3 \int_0^1 \theta_t^2 dx + \gamma \int_0^1 \Psi_x \theta_t dx - \int_0^1 (g \diamond \theta_x) \theta_x dx - \mu_2 \int_0^1 z(x, 1, t) \theta dx,$$

the last three terms can be estimated, using Young's inequality and Lemma 2.2, as follows, for  $\alpha > 0$ ,

$$\begin{aligned} \gamma \int_0^1 \Psi_x \theta_t dx & \leq \varepsilon_2 \int_0^1 \Psi_x^2 dx + \frac{\gamma^2}{4\varepsilon_2} \int_0^1 \theta_t^2 dx, \\ -\mu_2 \int_0^1 z(x, 1, t) \theta dx & \leq \alpha \int_0^1 \theta_x^2 dx + \frac{\mu_2^2 C_p}{4\alpha} \int_0^1 z^2(x, 1, t) dx \end{aligned}$$

and

$$- \int_0^1 (g \diamond \theta_x) \theta_x dx \leq \alpha \int_0^1 \theta_x^2 dx + \frac{1}{4\alpha} \int_0^t g(s) ds \int_0^1 (g \circ \theta_x) dx.$$

We deduce that

$$\begin{aligned} I_2'(t) \leq & - \left( \delta - \int_0^t g(s) ds - 2\alpha \right) \int_0^1 \theta_x^2 dx + \left( \rho_3 + \frac{\gamma^2}{4\varepsilon_2} \right) \int_0^1 \theta_t^2 dx \\ & + \varepsilon_2 \int_0^1 \Psi_x^2 dx + \frac{1}{4\alpha} \int_0^t g(s) ds \int_0^1 (g \circ \theta_x) dx + \frac{\mu_2^2 C_p}{4\alpha} \int_0^1 z^2(x, 1, t) dx. \end{aligned} \quad (4.9)$$

The choice of  $\alpha = \frac{\lambda}{4}$  gives the result.  $\square$

In order to get the negative terms of  $\int_0^1 \Phi_t^2 dx$  and  $\int_0^1 \Psi_t^2 dx$ , we define the functional

$$I_3(t) = -\rho_1 \int_0^1 \Phi_t \Phi dx - \rho_2 \int_0^1 \Psi_t \Psi dx.$$

**Lemma 4.4** *Let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2). Then we have*

$$\begin{aligned} I_3'(t) \leq & -\rho_1 \int_0^1 \Phi_t^2 dx - \rho_2 \int_0^1 \Psi_t^2 dx + \frac{3b}{2} \int_0^1 \Psi_x^2 dx \\ & + K \int_0^1 |\Phi_x + \Psi|^2 dx + \frac{\beta^2}{2b} \int_0^1 \theta_t^2 dx. \end{aligned} \quad (4.10)$$

**Proof.** A differentiation of  $I_3$ , taking into account (2.1)-(2.2), gives

$$I_3'(t) = -\rho_1 \int_0^1 \Phi_t^2 dx - \rho_2 \int_0^1 \Psi_t^2 dx + b \int_0^1 \Psi_x^2 dx + K \int_0^1 |\Phi_x + \Psi|^2 dx - \beta \int_0^1 \theta_t \Psi_x dx.$$

Using Young's and Poincaré's inequalities for the last term, we obtain (4.10).  $\square$

In order to get the negative term of  $\int_0^1 |\Phi_x + \Psi|^2 dx$ , we define the functional

$$\begin{aligned} I_4(t) = & \rho_2 \int_0^1 \Psi_t(\Phi_x + \Psi) dx + (\rho_2 + \gamma) \int_0^1 \Psi_x \Phi_t dx + \rho_3 \int_0^1 \theta_t \Phi_t dx \\ & + \left( \frac{K\rho_3}{\rho_1} + \beta \right) \int_0^1 \theta_x \Phi_x dx - \int_0^1 (g * \theta_x) \Phi_x dx. \end{aligned} \quad (4.11)$$

**Lemma 4.5** *Let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2) and assume that (2.9) holds. Then, for any  $\varepsilon_4 > 0$ , we get*

$$\begin{aligned} I_4'(t) \leq & \varepsilon_4 \left( 1 + \frac{\rho_3 K^2}{\rho_1^2 b} \right) [\Phi_x^2(1) + \Phi_x^2(0)] + \frac{b^2}{4\varepsilon_4} [\Psi_x^2(1) + \Psi_x^2(0)] + \frac{b\rho_3}{4\varepsilon_4} [\theta_t^2(1) + \theta_t^2(0)] \\ & - K \int_0^1 |\Phi_x + \Psi|^2 dx + \varepsilon_4 \int_0^1 \Psi_x^2 dx + \frac{\delta^2 + \mu_1^2}{4\varepsilon_4} \int_0^1 \theta_t^2 dx + 2\varepsilon_4 \int_0^1 \Phi_t^2 dx \\ & + \rho_2 \int_0^1 \Psi_t^2 dx + \varepsilon_4 \int_0^1 \Phi_x^2 dx + \frac{g^2(0)}{2\varepsilon_4} \int_0^1 \theta_x^2 dx - \frac{g(0)}{2\varepsilon_4} \int_0^1 (g' \circ \theta_x) dx \\ & + \frac{\mu_2^2}{4\varepsilon_4} \int_0^1 z^2(x, 1, t) dx. \end{aligned} \quad (4.12)$$

**Proof.** By differentiating the functional  $I_4$ , using Eq. (2.1)<sub>1</sub>, (2.1)<sub>2</sub> and (2.1)<sub>3</sub>, we obtain

$$\begin{aligned} I_4'(t) = & \int_0^1 [b\Psi_{xx} - K(\Phi_x + \Psi) - \beta\theta_{tx}] (\Phi_x + \Psi) dx + \rho_2 \int_0^1 \Psi_t (\Phi_x + \Psi)_t dx \\ & + (\rho_2 + \gamma) \int_0^1 \Psi_{tx} \Phi_t dx + (\rho_2 + \gamma) \int_0^1 \Psi_x \Phi_{tt} dx + \frac{K\rho_3}{\rho_1} \int_0^1 \theta_t (\Phi_x + \Psi)_x dx \\ & + \int_0^1 (\delta\theta_{xx} - \gamma\Psi_{tx} - \mu_1\theta_t - \mu_2z(x, 1, t) - g * \theta_{xx}) \Phi_t dx + \left( \frac{K\rho_3}{\rho_1} + \beta \right) \int_0^1 \theta_{tx} \Phi_x dx \\ & + \left( \frac{K\rho_3}{\rho_1} + \beta \right) \int_0^1 \theta_x \Phi_{tx} dx - \int_0^1 (g * \theta_x)_t \Phi_x dx - \int_0^1 (g * \theta_x) \Phi_{tx} dx \\ = & b[\Phi_x \Psi_x]_{x=0}^{x=1} - \frac{b\rho_1}{K} \int_0^1 \Psi_x \Phi_{tt} dx - K \int_0^1 |\Phi_x + \Psi|^2 dx - \beta \int_0^1 \theta_{tx} \Phi_x dx + \beta \int_0^1 \theta_t \Psi_x dx \\ & + \rho_2 \int_0^1 \Psi_t \Phi_{tx} dx + \rho_2 \int_0^1 \Psi_t^2 dx + (\rho_2 + \gamma) \int_0^1 \Psi_{tx} \Phi_t dx + (\rho_2 + \gamma) \int_0^1 \Psi_x \Phi_{tt} dx \\ & + \frac{K\rho_3}{\rho_1} \int_0^1 \theta_t \Psi_x dx + \frac{K\rho_3}{\rho_1} [\Phi_x \theta_t]_{x=0}^{x=1} - \frac{K\rho_3}{\rho_1} \int_0^1 \theta_{tx} \Phi_x dx - \delta \int_0^1 \theta_x \Phi_{tx} dx - \gamma \int_0^1 \Psi_{tx} \Phi_t dx \\ & - \int_0^1 (\mu_1\theta_t + \mu_2z(x, 1, t)) \Phi_t dx + \int_0^1 (g * \theta_x) \Phi_{tx} dx + \left( \frac{K\rho_3}{\rho_1} + \beta \right) \int_0^1 \theta_{tx} \Phi_x dx \\ & + \left( \frac{K\rho_3}{\rho_1} + \beta \right) \int_0^1 \theta_x \Phi_{tx} dx - \int_0^1 (g * \theta_x)_t \Phi_x dx - \int_0^1 (g * \theta_x) \Phi_{tx} dx \end{aligned}$$

$$\begin{aligned}
&= b[\Phi_x \Psi_x]_{x=0}^{x=1} + \frac{K\rho_3}{\rho_1}[\Phi_x \theta_t]_{x=0}^{x=1} - K \int_0^1 |\Phi_x + \Psi|^2 dx + \left( \beta + \frac{K\rho_3}{\rho_1} \right) \int_0^1 \theta_t \Psi_x dx + \rho_2 \int_0^1 \Psi_t^2 dx \\
&\quad - \int_0^1 (\mu_1 \theta_t + \mu_2 z(x, 1, t)) \Phi_t dx - \int_0^1 (g * \theta_x)_t \Phi_x dx + \left( \rho_2 + \gamma - \frac{b\rho_1}{K} \right) \int_0^1 \Psi_x \Phi_{tt} dx \\
&\quad + \left( \frac{K\rho_3}{\rho_1} + \beta - \delta \right) \int_0^1 \theta_x \Phi_{tx} dx. \tag{4.13}
\end{aligned}$$

By using (2.9), we have

$$\begin{aligned}
I'_4(t) &= b[\Phi_x \Psi_x]_{x=0}^{x=1} + \frac{K\rho_3}{\rho_1}[\Phi_x \theta_t]_{x=0}^{x=1} - K \int_0^1 |\Phi_x + \Psi|^2 dx + \delta \int_0^1 \theta_t \Psi_x dx \\
&\quad + \rho_2 \int_0^1 \Psi_t^2 dx - \int_0^1 (\mu_1 \theta_t + \mu_2 z(x, 1, t)) \Phi_t dx - \int_0^1 (g * \theta_x)_t \Phi_x dx. \tag{4.14}
\end{aligned}$$

Now we estimate the terms in the right side of (4.14). Applying Young's and Poincaré's inequalities and Lemma 2.2, we obtain that for any  $\varepsilon_4 > 0$ ,

$$\begin{aligned}
&\delta \int_0^1 \theta_t \Psi_x dx \leq \varepsilon_4 \int_0^1 \Psi_x^2 dx + \frac{\delta^2}{4\varepsilon_4} \int_0^1 \theta_t^2 dx, \\
&-\mu_1 \int_0^1 \theta_t \Phi_t dx \leq \varepsilon_4 \int_0^1 \Phi_t^2 dx + \frac{\mu_1^2}{4\varepsilon_4} \int_0^1 \theta_t^2 dx, \\
&-\mu_2 \int_0^1 z(x, 1, t) \Phi_t dx \leq \varepsilon_4 \int_0^1 \Phi_t^2 dx + \frac{\mu_2^2}{4\varepsilon_4} \int_0^1 z^2(x, 1, t) dx, \\
&b[\Phi_x \Psi_x]_{x=0}^{x=1} \leq \varepsilon_4 [\Phi_x^2(1) + \Phi_x^2(0)] + \frac{b^2}{4\varepsilon_4} [\Psi_x^2(1) + \Psi_x^2(0)], \\
&\frac{K\rho_3}{\rho_1} [\Phi_x \theta_t]_{x=0}^{x=1} \leq \frac{\varepsilon_4 \rho_3 K^2}{\rho_1^2 b} [\Phi_x^2(1) + \Phi_x^2(0)] + \frac{b\rho_3}{4\varepsilon_4} [\theta_t^2(1) + \theta_t^2(0)]
\end{aligned}$$

and

$$\begin{aligned}
&-\int_0^1 (g * \theta_x)_t \Phi_x dx = -\int_0^1 (g(0)\theta_x + g' * \theta_x) \Phi_x dx = -\int_0^1 (g(t)\theta_x - g' \diamond \theta_x) \Phi_x dx \\
&\leq -\frac{g(0)}{2\varepsilon_4} \int_0^1 (g' \diamond \theta_x) dx + \frac{\varepsilon_4}{2} \int_0^1 \Phi_x^2 dx + \frac{\varepsilon_4}{2} \int_0^1 \Phi_x^2 dx + \frac{g^2(0)}{2\varepsilon_4} \int_0^1 \theta_x^2 dx. \tag{4.15}
\end{aligned}$$

Combining all the above estimates, we get the desired results.  $\square$

In order to absorb the boundary terms, appearing in (4.12), we exploit as in [17], the following function:

$$q(x) = 2 - 4x, \quad x \in [0, 1].$$

We will also introduce the functionals  $J_1$  and  $J_2$  defined by

$$J_1(t) = \rho_1 \int_0^1 \Phi_t q \Phi_x dx \tag{4.16}$$

and

$$J_2(t) = \gamma \rho_2 b \int_0^1 \Psi_t q \Psi_x dx + \frac{\beta b}{\delta} \int_0^1 (\rho_3 \theta_t + \gamma \Psi_x) q (\delta \theta_x - (g * \theta_x)) dx. \tag{4.17}$$

**Lemma 4.6** *Let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2). For any  $\varepsilon_4 > 0$ , we have*

$$J'_1(t) \leq -K[\Phi_x^2(1) + \Phi_x^2(0)] + 2\rho_1 \int_0^1 \Phi_t^2 dx + 3K \int_0^1 \Phi_x^2 dx + K \int_0^1 \Psi_x^2 dx \quad (4.18)$$

and

$$\begin{aligned} J'_2(t) &\leq -b^2\gamma[\Psi_x^2(1) + \Psi_x^2(0)] - \beta b\rho_3[\theta_t^2(1) + \theta_t^2(0)] + \varepsilon_4^2 K^2 \int_0^1 |\Phi_x + \Psi|^2 dx + 2\gamma\rho_2 b \int_0^1 \Psi_t^2 dx \\ &\quad + \left(2b^2\gamma + \frac{\gamma^2 b^2}{4\varepsilon_4^2} + \varepsilon_4\right) \int_0^1 \Psi_x^2 dx + \left(2\beta b\rho_3 + \frac{5\varepsilon_4}{2}\right) \int_0^1 \theta_t^2 dx + C_1(\varepsilon_4) \int_0^1 \theta_x^2 dx \\ &\quad + \frac{3\varepsilon_4}{2} \int_0^1 z^2(x, 1, t) dx + \frac{2\beta b \int_0^t g(s) ds}{\delta^2 \varepsilon_4} [4\delta\varepsilon_4 + \beta b\mu_1^2 + \beta b\mu_2^2] \int_0^1 (g \circ \theta_x) dx \\ &\quad - \frac{2g(0)\beta^2 b^2 (\gamma^2 + \rho_3^2)}{\varepsilon_4 \delta^2} \int_0^1 (g' \circ \theta_x) dx, \end{aligned} \quad (4.19)$$

where

$$C_1(\varepsilon_4) = \frac{4\beta b}{\delta} \left[ 2 \left( \int_0^t g(s) ds \right)^2 + \delta^2 \right] + \frac{2\beta^2 b^2}{\varepsilon_4} \left[ \frac{g^2(0)(\gamma^2 + \rho_3^2)}{\delta^2} + (\mu_1^2 + \mu_2^2) \left( 1 + \frac{\left( \int_0^t g(s) ds \right)^2}{\delta^2} \right) \right].$$

**Proof.** A direct differentiation of  $J_1$  yields

$$\begin{aligned} J'_1(t) &= K \int_0^1 (\Phi_x + \Psi)_x q \Phi_x dx + \rho_1 \int_0^1 \Phi_t q \Phi_{tx} dx \\ &= K \int_0^1 \Phi_{xx} q \Phi_x dx + K \int_0^1 \Psi_x q \Phi_x dx - \frac{\rho_1}{2} \int_0^1 q_x \Phi_t^2 dx \\ &= \frac{K}{2} [q \Phi_x^2]_{x=0}^{x=1} - \frac{K}{2} \int_0^1 q_x \Phi_x^2 dx + K \int_0^1 \Psi_x q \Phi_x dx + 2\rho_1 \int_0^1 \Phi_t^2 dx \\ &= -K[\Phi_x^2(1) + \Phi_x^2(0)] + 2K \int_0^1 \Phi_x^2 dx + 2\rho_1 \int_0^1 \Phi_t^2 dx + K \int_0^1 \Psi_x q \Phi_x dx. \end{aligned} \quad (4.20)$$

The Young's inequality applied to the last term gives the result.

Differentiating  $J_2(t)$  along solutions of (2.1), we find

$$\begin{aligned} J'_2(t) &= \gamma b \int_0^1 [b\Psi_{xx} - K(\Phi_x + \Psi) - \beta\theta_{tx}] q \Psi_x dx + \gamma\rho_2 b \int_0^1 \Psi_t q \Psi_{tx} dx \\ &\quad + \frac{\beta b}{\delta} \int_0^1 [(\delta\theta_{xx} - \gamma\Psi_{tx} - \mu_1\theta_t - \mu_2 z(x, 1, t) - g * \theta_{xx}) + \gamma\Psi_{tx}] q (\delta\theta_x - g * \theta_x) dx \\ &\quad + \frac{\beta b}{\delta} \int_0^1 (\rho_3\theta_t + \gamma\Psi_x) q (\delta\theta_{tx} - (g * \theta_x)_t) dx. \end{aligned} \quad (4.21)$$

By integration by part, we obtain

$$\begin{aligned} J'_2(t) &= \frac{b^2\gamma}{2} [q\Psi_x^2]_{x=0}^{x=1} - \frac{b^2\gamma}{2} \int_0^1 q_x \Psi_x^2 dx - K\gamma b \int_0^1 (\Phi_x + \Psi) q \Psi_x dx \\ &\quad - \beta\gamma b \int_0^1 \theta_{tx} q \Psi_x dx - \frac{\gamma\rho_2 b}{2} \int_0^1 q_x \Psi_t^2 dx - \frac{\beta b}{2\delta} \int_0^1 q_x (\delta\theta_x - (g * \theta_x))^2 dx \end{aligned}$$



$$\begin{aligned}
& -\frac{\beta b \mu_1}{\delta} \int_0^1 \theta_t q (\delta \theta_x - (g * \theta_x)) dx - \frac{\beta b \mu_2}{\delta} \int_0^1 z(x, 1, t) q (\delta \theta_x - (g * \theta_x)) dx + \frac{\beta b \rho_3}{2} [q \theta_t^2]_{x=0}^{x=1} \\
& -\frac{\beta b \rho_3}{2} \int_0^1 q_x \theta_t^2 dx - \frac{\beta b \rho_3}{\delta} \int_0^1 \theta_t q (g * \theta_x)_t dx + \gamma \beta b \int_0^1 \Psi_x q \theta_{tx} dx - \frac{b \beta \gamma}{\delta} \int_0^1 \Psi_x q (g * \theta_x)_t dx \\
& = -b^2 \gamma [\Psi_x^2(1) + \Psi_x^2(0)] + 2b^2 \gamma \int_0^1 \Psi_x^2 dx - K \gamma b \int_0^1 (\Phi_x + \Psi) q \Psi_x dx + 2\gamma \rho_2 b \int_0^1 \Psi_t^2 dx \\
& -\frac{\beta b \gamma}{\delta} \int_0^1 \Psi_x q (g * \theta_x)_t dx - \beta b \mu_1 \int_0^1 \theta_t q \theta_x dx + \frac{\beta b \mu_1}{\delta} \int_0^1 \theta_t q (g * \theta_x) dx + 2\beta b \rho_3 \int_0^1 \theta_t^2 dx \\
& -\beta b \mu_2 \int_0^1 z(x, 1, t) q \theta_x dx + \frac{\beta b \mu_2}{\delta} \int_0^1 z(x, 1, t) q (g * \theta_x) dx - \beta b \rho_3 [\theta_t^2(1) + \theta_t^2(0)] \\
& -\frac{\beta b \rho_3}{\delta} \int_0^1 \theta_t q (g * \theta_x)_t dx + \frac{2\beta b}{\delta} \int_0^1 (\delta \theta_x - (g * \theta_x))^2 dx. \tag{4.22}
\end{aligned}$$

Note that

$$\begin{aligned}
& -\frac{\beta b \gamma}{\delta} \int_0^1 \Psi_x q (g * \theta_x)_t dx = -\frac{\beta b \gamma}{\delta} \int_0^1 \Psi_x q (g(t) \theta_x - g' \diamond \theta_x) dx \\
& \leq \frac{2}{\varepsilon_4} \left( \frac{b \beta \gamma}{\delta} \right)^2 g^2(0) \int_0^1 \theta_x^2 dx + \varepsilon_4 \int_0^1 \Psi_x^2 dx - \frac{2}{\varepsilon_4} \left( \frac{b \beta \gamma}{\delta} \right)^2 g(0) \int_0^1 (g' \diamond \theta_x) dx \tag{4.23}
\end{aligned}$$

and similarly

$$-\frac{\beta b \rho_3}{\delta} \int_0^1 \theta_t q (g * \theta_x)_t dx \leq \frac{2}{\varepsilon_4} \left( \frac{b \beta \rho_3}{\delta} \right)^2 g^2(0) \int_0^1 \theta_x^2 dx + \varepsilon_4 \int_0^1 \theta_t^2 dx - \frac{2}{\varepsilon_4} \left( \frac{b \beta \rho_3}{\delta} \right)^2 g(0) \int_0^1 (g' \diamond \theta_x) dx.$$

Moreover,

$$\begin{aligned}
& -\beta b \mu_1 \int_0^1 \theta_t q \theta_x dx \leq \frac{\varepsilon_4}{2} \int_0^1 \theta_t^2 dx + \frac{2\beta^2 b^2 \mu_1^2}{\varepsilon_4} \int_0^1 \theta_x^2 dx, \\
& -\beta b \mu_2 \int_0^1 z(x, 1, t) q \theta_x dx \leq \frac{\varepsilon_4}{2} \int_0^1 z^2(x, 1, t) dx + \frac{2\beta^2 b^2 \mu_2^2}{\varepsilon_4} \int_0^1 \theta_x^2 dx, \\
& -K \gamma b \int_0^1 (\Phi_x + \Psi) q \Psi_x dx \leq \varepsilon_4^2 K^2 \int_0^1 |\Phi_x + \Psi|^2 dx + \frac{\gamma^2 b^2}{4\varepsilon_4^2} \int_0^1 \Psi_x^2 dx, \\
& \frac{2\beta b}{\delta} \int_0^1 (\delta \theta_x - (g * \theta_x))^2 dx \leq \frac{8\beta b}{\delta} \left( \int_0^t g(s) ds \right)^2 \int_0^1 \theta_x^2 dx + 4\beta b \delta \int_0^1 \theta_x^2 dx \\
& \quad + \frac{8\beta b}{\delta} \left( \int_0^t g(s) ds \right) \int_0^1 (g \diamond \theta_x) dx \tag{4.24}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\beta b \mu_1}{\delta} \int_0^1 \theta_t q (g * \theta_x) dx \leq \frac{\varepsilon_4}{2} \int_0^1 \theta_t^2 dx + \frac{2}{\varepsilon_4} \left( \frac{\beta b \mu_1}{\delta} \right)^2 \int_0^t g(s) ds \int_0^1 (g \diamond \theta_x) dx \\
& \quad + \frac{\varepsilon_4}{2} \int_0^1 \theta_t^2 dx + \frac{2}{\varepsilon_4} \left( \frac{\beta b \mu_1}{\delta} \right)^2 \left( \int_0^t g(s) ds \right)^2 \int_0^1 \theta_x^2 dx, \tag{4.25}
\end{aligned}$$

$$\frac{\beta b \mu_2}{\delta} \int_0^1 z(x, 1, t) q (g * \theta_x) dx \leq \varepsilon_4 \int_0^1 z^2(x, 1, t) dx + \frac{2}{\varepsilon_4} \left( \frac{\beta b \mu_2}{\delta} \right)^2 \int_0^t g(s) ds \int_0^1 (g \diamond \theta_x) dx$$

$$+\frac{2}{\varepsilon_4} \left( \frac{\beta b \mu_2}{\delta} \right)^2 \left( \int_0^t g(s) ds \right)^2 \int_0^1 \theta_x^2 dx. \quad (4.26)$$

This completes the proof of the lemma.  $\square$

Now, let us define the following functional

$$I_5(t) = \rho_2 \rho_3 \int_0^1 \int_0^x \theta_t(t, y) dy \Psi_t dx.$$

**Lemma 4.7** *Let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2). For any  $\varepsilon_2 > 0$  and  $\eta_1 > 0$ , we have*

$$\begin{aligned} I_5'(t) &\leq -\frac{\rho_2 \gamma}{4} \int_0^1 \Psi_t^2 dx + \frac{\rho_2}{\gamma} \left[ \delta^2 + 2 \left( \int_0^t g(s) ds \right)^2 \right] \int_0^1 \theta_x^2 dx + \frac{2\rho_2 \mu_2^2}{\gamma} \int_0^1 z^2(x, 1, t) dx \\ &\quad + \varepsilon_2 (C_p + 1) \int_0^1 \Psi_x^2 dx + \frac{2\rho_2}{\gamma} \int_0^t g(s) ds \int_0^1 (g \circ \theta_x) dx + \frac{b^2}{4\eta_1} \Psi_x^2(1) + \frac{\rho_3 b}{4\eta_1} \theta_t^2(1) \\ &\quad + \left[ \beta \rho_3 + \frac{\rho_2 \mu_1^2}{\gamma} + \frac{\rho_3^2}{4\varepsilon_2} (2K^2 + b^2) + \eta_1 \left( \rho_3^2 + \frac{\rho_3 \beta^2}{b} \right) \right] \int_0^1 \theta_t^2 dx \\ &\quad + \varepsilon_2 C_p \int_0^1 \Phi_x^2 dx. \end{aligned} \quad (4.27)$$

**Proof.** Differentiating the functional  $I_5$  and using Eqs. (2.1), we obtain

$$\begin{aligned} I_5'(t) &= \rho_2 \int_0^1 \int_0^x [\delta \theta_{xx} - \gamma \Psi_{tx} - \mu_1 \theta_t - \mu_2 z(y, 1, t) - g * \theta_{xx}] dy \Psi_t dx \\ &\quad + \rho_3 \int_0^1 \int_0^x \theta_t dy [b \Psi_{xx} - K(\Phi_x + \Psi) - \beta \theta_{tx}] dx \\ &= -\gamma \rho_2 \int_0^1 \Psi_t^2 dx + \rho_2 \delta \int_0^1 \theta_x \Psi_t dx - \rho_2 \int_0^1 (g * \theta_x) \Psi_t dx - \mu_1 \rho_2 \int_0^1 \int_0^x \theta_t dy \Psi_t dx \\ &\quad - \mu_2 \rho_2 \int_0^1 \int_0^x z(y, 1, t) dy \Psi_t dx - K \rho_3 \int_0^1 \int_0^x \theta_t dy \Psi dx - \rho_3 b \int_0^1 \theta_t \Psi_x dx \\ &\quad + K \rho_3 \int_0^1 \theta_t \Phi dx + \beta \rho_3 \int_0^1 \theta_t^2 dx + \rho_3 \left[ \int_0^x \theta_t dy (b \Psi_x - \beta \theta_t) \right]_{x=0}^{x=1}. \end{aligned} \quad (4.28)$$

Using Young's and Poincaré's inequalities, we find

$$\begin{aligned} I_5'(t) &\leq -\frac{\gamma \rho_2}{2} \int_0^1 \Psi_t^2 dx + \frac{\rho_2 \delta^2}{\gamma} \int_0^1 \theta_x^2 dx + \varepsilon_2 (1 + C_p) \int_0^1 \Psi_x^2 dx + \varepsilon_2 C_p \int_0^1 \Phi_x^2 dx \\ &\quad + \left[ \beta \rho_3 + \frac{2\rho_2 \mu_1^2}{\gamma} + \frac{\rho_3^2}{4\varepsilon_2} (2K^2 + b^2) \right] \int_0^1 \theta_t^2 dx + \frac{2\rho_2 \mu_2^2}{\gamma} \int_0^1 z^2(x, 1, t) dx \\ &\quad - \rho_2 \int_0^1 (g * \theta_x) \Psi_t dx + \rho_3 \left[ \int_0^x \theta_t dy (b \Psi_x - \beta \theta_t) \right]_{x=0}^{x=1}. \end{aligned} \quad (4.29)$$

We can estimate terms in the right side as follows

$$\rho_2 \int_0^1 (g * \theta_x) \Psi_t dx = -\rho_2 \int_0^1 (g \diamond \theta_x) \Psi_t dx + \rho_2 \int_0^t g(s) ds \int_0^1 \theta_x \Psi_t dx$$

$$\leq \frac{\rho_2 \gamma}{4} \int_0^1 \Psi_t^2 dx + \frac{2\rho_2}{\gamma} \int_0^t g(s) ds \int_0^1 (g \circ \theta_x) dx + \frac{2\rho_2}{\gamma} \left( \int_0^t g(s) ds \right)^2 \int_0^1 \theta_x^2 dx \quad (4.30)$$

and

$$\rho_3 \left[ \int_0^x \theta_t^2 dy (b\Psi_x - \beta\theta_t) \right]_{x=0}^{x=1} \leq \frac{b^2}{4\eta_1} \Psi_x^2(1) + \frac{\rho_3 b}{4\eta_1} \theta_t^2(1) + \eta_1 \left( \rho_3^2 + \frac{\rho_3 \beta^2}{b} \right) \int_0^1 \theta_t^2 dx.$$

The proof is completed.  $\square$

Finally, as in [6], we introduce the functional

$$I_6(t) = \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t) d\rho dx.$$

Then the following result holds:

**Lemma 4.8** *Let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2). Then, we have*

$$I_6'(t) \leq -2I_6(t) - \frac{c}{\tau} \int_0^1 z^2(x, 1, t) dx + \frac{1}{\tau} \int_0^1 \theta_t^2 dx. \quad (4.31)$$

where  $c$  is a positive constant.

**Proof.** Differentiating the functional  $I_6$ , we have

$$\begin{aligned} I_6'(t) &= -\frac{2}{\tau} \int_0^1 \int_0^1 e^{-2\tau\rho} z z_\rho(x, \rho, t) d\rho dx \\ &= -2 \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t) d\rho dx - \frac{1}{\tau} \int_0^1 \int_0^1 \frac{\partial}{\partial \rho} (e^{-2\tau\rho} z^2(x, \rho, t)) d\rho dx \\ &= -2I_6(t) + \frac{1}{\tau} \int_0^1 \theta_t^2 dx - \frac{1}{\tau} \int_0^1 e^{-2\tau} z^2(x, 1, t) dx. \end{aligned} \quad (4.32)$$

The above equality implies that there exists a positive constant  $c$  such that (4.31) holds.  $\square$

Now, we define the Lyapunov functional  $\mathcal{L}$  as follows

$$\begin{aligned} \mathcal{L}(t) &= NE(t) + N_1 I_1(t) + N_2 I_2(t) + \frac{v}{4} I_3(t) + v I_4(t) + N_5 I_5(t) + I_6(t) \\ &\quad + v\varepsilon_4 \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) J_1(t) + \frac{1}{2\varepsilon_4} J_2(t), \quad t \geq 0, \end{aligned} \quad (4.33)$$

where  $N, N_1, N_2, N_5$  are positive constants to be chosen properly later and  $v = \min\{\gamma, \beta\}$ . For large  $N$ , we can verify that, for some  $m, M > 0$ ,

$$mE(t) \leq \mathcal{L}(t) \leq ME(t), \quad t \geq 0. \quad (4.34)$$

Taking into account (4.1), (4.7), (4.8), (4.10), (4.12), (4.18), (4.19), (4.27), (4.31) and the relations

$$\int_0^1 \Phi_x^2 dx \leq 2 \int_0^1 |\Phi_x + \Psi|^2 dx + 2C_p \int_0^1 \Psi_x^2 dx, \quad (4.35)$$

we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \left\{ \frac{\rho_1 v}{4} - N_1 \varepsilon_1 - 2\varepsilon_4 v \left[ 1 + \rho_1 \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) \right] \right\} \int_0^1 \Phi_t^2 dx \\
& - \left\{ \frac{\rho_2 \gamma N_5}{4} - N_1 \left( \frac{3\rho_2}{2} + \frac{\rho_1^2 C_p}{4\varepsilon_1} \right) - \frac{3\rho_2 v}{4} - \frac{\gamma \rho_2 b}{\varepsilon_4} \right\} \int_0^1 \Psi_t^2 dx \\
& - \left\{ \frac{3vK}{4} - 2\varepsilon_2 C_p N_5 - \varepsilon_4 \left[ 6vK \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) + 2v + \frac{K^2}{2} \right] \right\} \int_0^1 |\Phi_x + \Psi|^2 dx \\
& - \left\{ N_1 b - \varepsilon_2 (N_2 + N_5 (1 + C_p + 2C_p^2)) - \frac{1}{2\varepsilon_4} \left[ 2b^2 \gamma + \frac{\gamma^2 b^2}{4\varepsilon_4^2} + \varepsilon_4 \right] \right. \\
& \quad \left. - \varepsilon_4 (Kv + 6C_p Kv) \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) - \frac{3bv}{8} - \varepsilon_4 (v + 2vC_p) \right\} \int_0^1 \Psi_x^2 dx \\
& - \left\{ Nm_0 - N_2 \left( \rho_3 + \frac{\gamma^2}{4\varepsilon_2} \right) - \left[ \frac{v}{4\varepsilon_4} (\delta^2 + \mu_1^2) + \frac{\beta b \rho_3}{\varepsilon_4} + \frac{5}{4} \right] - \frac{\beta^2 v}{8b} \right. \\
& \quad \left. - N_5 \left[ \beta \rho_3 + \frac{\rho_2 \mu_1^2}{\gamma} + \frac{\rho_3^2}{4\varepsilon_2} (2K^2 + b^2) + \eta_1 \left( \rho_3^2 + \frac{\rho_3 \beta^2}{b} \right) \right] - \frac{1}{\tau} \right\} \int_0^1 \theta_t^2 dx \\
& - \left\{ \frac{\lambda N_2}{2} - \frac{N_1 \beta^2}{2\rho_2} - \frac{1}{2\varepsilon_4} (vg^2(0) + C_1(\varepsilon_4)) - \frac{N_5 \rho_2}{\gamma} (\delta^2 + 2\bar{g}^2) \right\} \int_0^1 \theta_x^2 dx \\
& + \left\{ \frac{N_2 \bar{g}}{\lambda} + \frac{\beta b \bar{g}}{\varepsilon_4^2 \delta^2} (4\delta\varepsilon_4 + \beta b \mu_1^2 + \beta b \mu_2^2) + \frac{2\rho_2 \bar{g} N_5}{\gamma} \right\} \int_0^1 (g \circ \theta_x) dx \\
& + \left\{ \frac{\beta N}{2} - \frac{vg(0)}{2\varepsilon_4} - \frac{g(0)b^2 \beta^2 (\gamma^2 + \rho_3^2)}{\varepsilon_4^2 \delta^2} \right\} \int_0^1 (g' \circ \theta_x) dx - 2I_6(t) \\
& - \left\{ Nm_0 + \frac{c}{\tau} - \frac{\mu_2^2 N_2 C_p}{\lambda} - \frac{\mu_2^2 v}{4\varepsilon_4} - \frac{3}{4} - \frac{2\rho_2 \mu_2^2 N_5}{\gamma} \right\} \int_0^1 z^2(x, 1, t) dx, \tag{4.36}
\end{aligned}$$

where  $m_0 = \min \left\{ \beta \left( \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right), \beta \left( \frac{\xi}{2\tau} - \frac{\mu_2}{2} \right) \right\}$ . At this point, we need to choose our constants very carefully. First, let us pick  $\eta_1 = \frac{N_5 \varepsilon_4}{v}$  and choose

$$\varepsilon_4 \leq \min \left\{ \frac{\rho_1}{16} \left[ 1 + \rho_1 \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) \right]^{-1}, \frac{3vK}{8} \left[ 6vK \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) + 2v + \frac{K^2}{2} \right]^{-1} \right\}. \tag{4.37}$$

Second, we select  $N_1$  sufficiently large such that

$$\begin{aligned}
\frac{N_1 b}{2} \geq & \frac{1}{2\varepsilon_4} \left[ 2b^2 \gamma + \frac{\gamma^2 b^2}{4\varepsilon_4^2} + \varepsilon_4 \right] + \varepsilon_4 Kv (1 + 6C_p) \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) \\
& + \frac{3bv}{8} + \varepsilon_4 v (1 + 2C_p), \tag{4.38}
\end{aligned}$$

then we choose  $\varepsilon_1$  so small that

$$\varepsilon_1 \leq \frac{\rho_1 v}{16N_1}.$$

Next, we choose  $N_5$  sufficiently large so that

$$N_5 \geq \frac{8}{\rho_2 \gamma} \left[ N_1 \left( \frac{3\rho_2}{2} + \frac{\rho_1^2 C_p}{4\varepsilon_1} \right) + \frac{\gamma \rho_2 b}{\varepsilon_4} + \frac{3\rho_2 v}{4} \right],$$

and also select  $N_2$  sufficiently large so that

$$\frac{\lambda N_2}{4} > \frac{N_1 \beta^2}{2\rho_2} + \frac{1}{2\varepsilon_4} (vg^2(0) + C_1(\varepsilon_4)) + \frac{N_5 \rho_2}{\gamma} (\delta^2 + 2\bar{g}^2).$$

Furthermore, we select  $\varepsilon_2$  satisfies

$$\varepsilon_2 < \min \left\{ \frac{N_1 b}{2(N_2 + N_5(1 + C_p + 2C_p^2))}, \frac{3Kv}{16N_5 C_p} \right\}.$$

Finally, we choose  $N$  large enough so that (4.34) remains valid and (4.36) takes the form

$$\begin{aligned} \mathcal{L}'(t) &\leq -C_1 \int_0^1 \left( \Phi_t^2 + \Psi_t^2 + |\Phi_x + \Psi|^2 + \Psi_x^2 + \theta_t^2 + \theta_x^2 + \int_0^1 z^2(x, \rho, t) d\rho \right) dx + C_2 \int_0^1 (g \circ \theta_x) dx \\ &\leq -CE(t) + C_3 \int_0^1 (g \circ \theta_x) dx, \end{aligned} \quad (4.39)$$

where  $C_1, C_2, C_3$ , and  $C$  are positive constants.

#### 4.2 The case $\mu_2 = \mu_1$

If  $\mu_1 = \mu_2 = \mu$ , then we can choose  $\xi = \tau\mu$  in (2.7) and Lemma 4.1 takes the form

**Lemma 4.9** *Let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2). Assume that  $\mu_1 = \mu_2 = \mu$  and  $g$  satisfies (H1) and (H2),  $\xi = \tau\mu$ . Then, the energy functional defined by (2.5) is a non-increasing function and it satisfies*

$$E'(t) \leq -\frac{\beta}{2} g(t) \int_0^1 \theta_x^2 dx + \frac{\beta}{2} \int_0^1 (g' \circ \theta_x) dx \leq 0, \quad t \geq 0. \quad (4.40)$$

The proof of Lemma 4.9 is an immediate consequence of Lemma 4.1, by choosing  $\xi = \tau\mu$ .

If  $\mu_1 = \mu_2 = \mu$ , we need some additional negative term of  $\int_0^1 \theta_t^2 dx$ . For this purpose, let us introduce the functional

$$I_7(t) = -\rho_3 \int_0^1 \theta_t (g \diamond \theta) dx.$$

Then, we have the following estimate:

**Lemma 4.10** *Let  $(\Phi, \Psi, \theta, z)$  be a solution of (2.1)-(2.2). Then for any  $\varepsilon_7 > 0$  and  $\eta_2 > 0$ , we have*

$$\begin{aligned} I_7'(t) &\leq -\left( \rho_3 \int_0^t g(s) ds - \eta_2 \right) \int_0^1 \theta_t^2 dx + \varepsilon_7 \int_0^1 \Psi_t^2 dx + \varepsilon_7 \left[ 1 + \left( \int_0^t g(s) ds \right)^2 \right] \int_0^1 \theta_x^2 dx \\ &\quad + \varepsilon_7 \int_0^1 z^2(x, 1, t) dx + C_2(\varepsilon_7) \int_0^1 (g \circ \theta_x) dx - \frac{\rho_3^2}{2\eta_2} g(0) C_p \int_0^1 (g' \circ \theta_x) dx, \end{aligned} \quad (4.41)$$

where

$$C_2(\varepsilon_7) = \frac{\int_0^t g(s) ds}{4\varepsilon_7} (\delta^2 + \gamma^2 + 4\varepsilon_7^2 + 2 + \mu_2^2 C_p) + \frac{\mu_1^2 C_p}{2\eta_2} \int_0^t g(s) ds.$$

**Proof.** A simple differentiation leads to

$$\begin{aligned}
I_7'(t) &= -\rho_3 \int_0^1 \theta_t (g \diamond \theta)_t dx - \rho_3 \int_0^1 \theta_{tt} (g \diamond \theta) dx \\
&= -\left( \rho_3 \int_0^t g(s) ds \right) \int_0^1 \theta_t^2 dx - \rho_3 \int_0^1 \theta_t (g' \diamond \theta) dx - \int_0^1 \left( \int_0^t g(t-s) \theta_x(s) ds \right) (g \diamond \theta_x) dx \\
&\quad + \delta \int_0^1 \theta_x (g \diamond \theta_x) dx - \gamma \int_0^1 \Psi_t (g \diamond \theta_x) dx + \mu_1 \int_0^1 \theta_t (g \diamond \theta) dx \\
&\quad + \mu_2 \int_0^1 z(x, 1, t) (g \diamond \theta) dx.
\end{aligned} \tag{4.42}$$

Terms in the right side of (4.42) are estimated as follows. Using Young's inequalities and Lemma 2.2, we obtain, for all  $\eta_2 > 0$ ,

$$\begin{aligned}
-\rho_3 \int_0^1 \theta_t (g' \diamond \theta) dx &\leq \frac{\eta_2}{2} \int_0^1 \theta_t^2 dx + \frac{\rho_3^2}{2\eta_2} \int_0^t (-g'(s) ds) \int_0^1 (-g' \diamond \theta) dx \\
&\leq \frac{\eta_2}{2} \int_0^1 \theta_t^2 dx - \frac{\rho_3^2}{2\eta_2} g(0) C_p \int_0^1 (g' \diamond \theta_x) dx.
\end{aligned} \tag{4.43}$$

Similarly, for any  $\varepsilon_7 > 0$ , we have

$$\begin{aligned}
\delta \int_0^1 \theta_x (g \diamond \theta_x) dx &\leq \varepsilon_7 \int_0^1 \theta_x^2 dx + \frac{\delta^2}{4\varepsilon_7} \int_0^t g(s) ds \int_0^1 (g \diamond \theta_x) dx, \\
-\gamma \int_0^1 \Psi_t (g \diamond \theta_x) dx &\leq \varepsilon_7 \int_0^1 \Psi_t^2 dx + \frac{\gamma^2}{4\varepsilon_7} \int_0^t g(s) ds \int_0^1 (g \diamond \theta_x) dx, \\
\mu_1 \int_0^1 \theta_t (g \diamond \theta) dx &\leq \frac{\eta_2}{2} \int_0^1 \theta_t^2 dx + \frac{\mu_1^2 C_p}{2\eta_2} \int_0^t g(s) ds \int_0^1 (g \diamond \theta_x) dx
\end{aligned}$$

and

$$\mu_2 \int_0^1 z(x, 1, t) (g \diamond \theta) dx \leq \varepsilon_7 \int_0^1 z^2(x, 1, t) dx + \frac{\mu_2^2 C_p}{4\varepsilon_7} \int_0^t g(s) ds \int_0^1 (g \diamond \theta_x) dx.$$

Finally,

$$\begin{aligned}
&-\int_0^1 \left( \int_0^t g(t-s) \theta_x(s) ds \right) (g \diamond \theta_x) dx \\
&\leq \frac{\varepsilon_7}{2} \int_0^1 \left( \int_0^t g(t-s) (\theta_x(t) - \theta_x(x) - \theta_x(t)) ds \right)^2 dx + \frac{1}{2\varepsilon_7} \int_0^1 (g \diamond \theta_x)^2 dx \\
&\leq \varepsilon_7 \left( \int_0^t g(s) ds \right)^2 \int_0^1 \theta_x^2 dx + \left( \varepsilon_7 + \frac{1}{2\varepsilon_7} \right) \int_0^1 (g \diamond \theta_x)^2 dx \\
&\leq \varepsilon_7 \left( \int_0^t g(s) ds \right)^2 \int_0^1 \theta_x^2 dx + \left( \varepsilon_7 + \frac{1}{2\varepsilon_7} \right) \int_0^t g(s) ds \int_0^1 (g \diamond \theta_x) dx.
\end{aligned} \tag{4.44}$$

Therefore, the assertion of the lemma follows by combining all the above estimates.  $\square$

Now, we define the following Lyapunov functional  $\mathcal{L}$  as:

$$\mathcal{L}(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + \frac{v}{4} I_3(t) + v I_4(t) + N_5 I_5(t) + N_6 I_6(t) + N_7 I_7(t)$$

$$+v\varepsilon_4 \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) J_1(t) + \frac{1}{2\varepsilon_4} J_2(t), \quad t \geq 0, \quad (4.45)$$

where  $N, N_1, N_2, N_5, N_6, N_7$  are positive real numbers which will be chosen later.

Since  $g$  is continuous and  $g(0) > 0$ , then for any  $t \geq t_0 > 0$ , we have

$$\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds = g_0.$$

Then, using the estimates (4.7), (4.8), (4.10), (4.12), (4.18), (4.19), (4.27), (4.31), (4.40), (4.41) and algebraic inequality (4.35), we get

$$\begin{aligned} \mathcal{L}'(t) \leq & - \left\{ \frac{\rho_1 v}{4} - N_1 \varepsilon_1 - 2\varepsilon_4 v \left[ 1 + \rho_1 \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) \right] \right\} \int_0^1 \Phi_t^2 dx - 2N_6 I_6(t) \\ & - \left\{ \frac{\lambda N_2}{2} - \frac{N_1 \beta^2}{2\rho_2} - N_7 \varepsilon_7 (1 + \bar{g}^2) - \frac{1}{2\varepsilon_4} (vg^2(0) + C_1(\varepsilon_4)) - \frac{N_5 \rho_2}{\gamma} (\delta^2 + 2\bar{g}^2) \right\} \int_0^1 \theta_x^2 dx \\ & - \left\{ N_7 (\rho_3 g_0 - \eta_2) - N_2 \left( \rho_3 + \frac{\gamma^2}{4\varepsilon_2} \right) - \left[ \frac{v}{4\varepsilon_4} (\delta^2 + \mu^2) + \frac{\beta b \rho_3}{\varepsilon_4} + \frac{5}{4} \right] - \frac{\beta^2 v}{8b} \right. \\ & - N_5 \left[ \beta \rho_3 + \frac{2\rho_2 \mu^2}{\gamma} + \frac{\rho_3^2}{4\varepsilon_2} (2K^2 + b^2) + \eta_1 \left( \rho_3^2 + \frac{\rho_3 \beta^2}{b} \right) \right] - \frac{N_6}{\tau} \left. \right\} \int_0^1 \theta_t^2 dx \\ & - \left\{ \frac{3vK}{4} - 2\varepsilon_2 C_p N_5 - \varepsilon_4 \left[ 6vK \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) + 2v + \frac{K^2}{2} \right] \right\} \int_0^1 |\Phi_x + \Psi|^2 dx \\ & - \left\{ \frac{\rho_2 \gamma N_5}{4} - N_1 \left( \frac{3\rho_2}{2} + \frac{\rho_1^2 C_p}{4\varepsilon_1} \right) - N_7 \varepsilon_7 - \frac{3\rho_2 v}{4} - \frac{\gamma \rho_2 b}{\varepsilon_4} \right\} \int_0^1 \Psi_t^2 dx \\ & - \left\{ N_1 b - \varepsilon_2 (N_2 + N_5 (1 + C_p + 2C_p^2)) - \frac{1}{2\varepsilon_4} \left[ 2b^2 \gamma + \frac{\gamma^2 b^2}{4\varepsilon_4^2} + \varepsilon_4 \right] \right. \\ & - \varepsilon_4 (Kv + 6C_p K v) \left( \frac{1}{K} + \frac{\rho_3 K}{\rho_1^2 b} \right) - \frac{3bv}{8} - \varepsilon_4 (v + 2vC_p) \left. \right\} \int_0^1 \Psi_x^2 dx \\ & + \left\{ \frac{N_2 \bar{g}}{2\lambda} + C_2(\varepsilon_7) + \frac{\beta b \bar{g}}{\varepsilon_4^2 \delta^2} (4\delta\varepsilon_4 + 2\beta b \mu^2) + \frac{2\rho_2 \bar{g} N_5}{\gamma} \right\} \int_0^1 (g \circ \theta_x) dx \\ & + \left\{ \frac{\beta N}{2} - \frac{\rho_3^2 g(0) C_p}{2\eta_2} N_7 - \frac{vg(0)}{2\varepsilon_4} - \frac{g(0)b^2 \beta^2 (\gamma^2 + \rho_3^2)}{\varepsilon_4^2 \delta^2} \right\} \int_0^1 (g' \circ \theta_x) dx \\ & - \left\{ \frac{N_6 c}{\tau} - \frac{\mu^2 N_2 C_p}{\lambda} - N_7 \varepsilon_7 - \frac{\mu^2 v}{4\varepsilon_4} - \frac{3}{4} - \frac{2\rho_2 \mu^2 N_5}{\gamma} \right\} \int_0^1 z^2(x, 1, t) dx. \end{aligned} \quad (4.46)$$

Now, our goal is to choose our constants in (4.46) in order to get the negative coefficients on the right-hand side of (4.46). To this end, let us pick  $\eta_1 = \frac{N_5 \varepsilon_4}{v}, \eta_2 = \frac{1}{4N_7}$  and we pick  $\varepsilon_4, N_1, \varepsilon_1, N_5, N_2, \varepsilon_2$  in the same order with the same values as the case  $\mu_2 < \mu_1$ , respectively.

Then we pick  $N_6$  large enough such that

$$\frac{N_6 c}{2\tau} > \frac{N_2 \mu^2 C_p}{\lambda} + \frac{\mu^2 v}{4\varepsilon_4} + \frac{3}{4} + \frac{2\rho_2 \mu^2 N_5}{\gamma}.$$

After that, we choose  $N_7$  sufficiently large so that

$$\frac{N_7 \rho_3 g_0}{2} - \frac{1}{8} > N_2 \left( \rho_3 + \frac{\gamma^2}{4\varepsilon_2} \right) + \frac{v(\delta^2 + \mu^2)}{4\varepsilon_4} + \frac{\beta b \rho_3}{\varepsilon_4} + \frac{5}{4} + \frac{N_6}{\tau} + \frac{\beta^2 v}{8b}$$

$$+N_5 \left[ \beta \rho_3 + \frac{2\rho_2 \mu^2}{\gamma} + \frac{\rho_3^2}{4\varepsilon_2} (2K^2 + b^2) + \eta_1 \left( \rho_3^2 + \frac{\rho_3 \beta^2}{b} \right) \right]. \quad (4.47)$$

Furthermore, choosing  $\varepsilon_7$  sufficiently small such that

$$\varepsilon_7 < \min \left\{ \frac{N_6 c}{2\tau N_7}, \frac{\rho_2 \gamma N_5}{8N_7}, \frac{\lambda N_2}{4N_7 (1 + \bar{g}^2)} \right\}.$$

Once all the above constants are fixed, we pick  $N$  large enough such that there exists two positive constants  $\hat{C}$  and  $\hat{C}_3$

$$\mathcal{L}'(t) \leq -\hat{C}E(t) + \hat{C}_3 \int_0^1 (g \circ \theta_x) dx, \quad t \geq t_0. \quad (4.48)$$

From (4.39) and (4.48), we can know that the Lyapunov functionals  $\mathcal{L}$  are of the same form under the two cases:  $\mu_2 < \mu_1$  and  $\mu_2 = \mu_1$

$$\mathcal{L}'(t) \leq -CE(t) + C_3 \int_0^1 (g \circ \theta_x) dx, \quad t \geq t_0. \quad (4.49)$$

**Continuity of the proof of Theorem 2.4.** Multiplying (4.49) by  $\zeta(t)$  gives

$$\zeta(t) \mathcal{L}'(t) \leq -C\zeta(t)E(t) + C_3\zeta(t) \int_0^1 (g \circ \theta_x) dx. \quad (4.50)$$

The last term can be estimated, using (H2), we obtain

$$\zeta(t) \int_0^1 (g \circ \theta_x) dx \leq - \int_0^1 (g' \circ \theta_x) dx \leq -\frac{2}{\beta} E'(t).$$

Thus, (4.50) becomes, for some positive constant  $C_4$ ,

$$\zeta(t) \mathcal{L}'(t) \leq -C\zeta(t)E(t) - C_4 E'(t). \quad (4.51)$$

It is clear that

$$L(t) = \zeta(t) \mathcal{L}(t) + C_4 E(t) \sim E(t).$$

Therefore, using (4.51) and the fact that  $\zeta'(t) \leq 0$ , we arrive at

$$L'(t) = \zeta'(t) \mathcal{L}(t) + \zeta(t) \mathcal{L}'(t) + C_4 E'(t) \leq -C\zeta(t)E(t). \quad (4.52)$$

A simple integration of (4.52) over  $(t_0, t)$  leads to

$$L(t) \leq L(t_0) e^{-C \int_{t_0}^t \zeta(s) ds}, \quad t \geq t_0. \quad (4.53)$$

Recalling (4.34), estimate (4.53) yields the desired result (2.10).  $\square$



## References

- [1] A. Djebabla and N. Tatar, Exponential stabilization of the Timoshenko system by a thermo-viscoelastic damping, *J. Dyn. Control Syst.* **16** (2010), no. 2, 189–210.
- [2] A. Djebabla and N. Tatar, Stabilization of the Timoshenko beam by thermal effect, *Mediterr. J. Math.* **7** (2010), no. 3, 373–385.
- [3] A. Djebabla and N. Tatar, Exponential stabilization of the Timoshenko system by a thermal effect with an oscillating kernel, *Math. Comput. Modelling* **54** (2011), no. 1-2, 301–314.
- [4] D. Feng, D. Shi and W. Zhang, Boundary feedback stabilization of Timoshenko beam with boundary dissipation, *Sci. China Ser. A* **41** (1998), no. 5, 483–490.
- [5] J. U. Kim and Y. Renardy, Boundary control of the Timoshenko beam, *SIAM J. Control Optim.* **25** (1987), no. 6, 1417–1429.
- [6] M. Kirane and B. Said-Houari, Existence and asymptotic stability of a viscoelastic wave equation with a delay, *Z. Angew. Math. Phys.* **62** (2011), no. 6, 1065–1082.
- [7] M. Kirane, B. Said-Houari and M. N. Anwar, Stability result for the Timoshenko system with a time-varying delay term in the internal feedbacks, *Commun. Pure Appl. Anal.* **10** (2011), no. 2, 667–686.
- [8] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.
- [9] W. Liu, General decay of the solution for a viscoelastic wave equation with a time-varying delay term in the internal feedback, *J. Math. Phys.* **54** (2013), no. 4, 043504, 9 pp.
- [10] Z. Liu and C. Peng, Exponential stability of a viscoelastic Timoshenko beam, *Adv. Math. Sci. Appl.* **8** (1998), no. 1, 343–351.
- [11] S. A. Messaoudi and A. Fareh, Energy decay in a Timoshenko-type system of thermoelasticity of type III with different wave-propagation speeds, *Arab. J. Math. (Springer)* **2** (2013), no. 2, 199–207.
- [12] S. A. Messaoudi, M. Pokojovy and B. Said-Houari, Nonlinear damped Timoshenko systems with second sound—global existence and exponential stability, *Math. Methods Appl. Sci.* **32** (2009), no. 5, 505–534.
- [13] S. A. Messaoudi and B. Said-Houari, Energy decay in a Timoshenko-type system of thermoelasticity of type III, *J. Math. Anal. Appl.* **348** (2008), no. 1, 298–307.

- [14] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.* **45** (2006), no. 5, 1561–1585 (electronic).
- [15] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Differential Integral Equations* **21** (2008), no. 9-10, 935–958.
- [16] C. A. Raposo et al., Exponential stability for the Timoshenko system with two weak dampings, *Appl. Math. Lett.* **18** (2005), no. 5, 535–541.
- [17] J. E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear Timoshenko systems-global existence and exponential stability, *J. Math. Anal. Appl.* **276** (2002), no. 1, 248–278.
- [18] J. E. Muñoz Rivera and R. Racke, Global stability for damped Timoshenko systems, *Discrete Contin. Dyn. Syst.* **9** (2003), no. 6, 1625–1639.
- [19] B. Said-Houari and Y. Laskri, A stability result of a Timoshenko system with a delay term in the internal feedback, *Appl. Math. Comput.* **217** (2010), no. 6, 2857–2869.
- [20] B. Said-Houari and R. Rahali, A stability result for a Timoshenko system with past history and a delay term in the internal feedback, *Dynam. Systems Appl.* **20** (2011), no. 2-3, 327–353.
- [21] M. de Lima Santos, Decay rates for solutions of a Timoshenko system with a memory condition at the boundary, *Abstr. Appl. Anal.* **7** (2002), no. 10, 531–546.
- [22] D.-H. Shi, S. H. Hou and D.-X. Feng, Feedback stabilization of a Timoshenko beam with an end mass, *Internat. J. Control* **69** (1998), no. 2, 285–300.
- [23] M. A. Shubov, Asymptotic and spectral analysis of the spatially nonhomogeneous Timoshenko beam model, *Math. Nachr.* **241** (2002), 125–162.
- [24] A. Soufyane and A. Wehbe, Uniform stabilization for the Timoshenko beam by a locally distributed damping, *Electron. J. Differential Equations* **2003**, No. 29, 14 pp. (electronic).
- [25] S. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Philos. Mag.* **41** (1921), 744–746.